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Finite Simple Groups
Exercise Sheet 1
Due 30.04.2019

Exercise 1 (6 Points)**.**
Let G be an abelian group of order n and suppose that it is isomorphic to $C_{m_1} \times \ldots \times C_{m_r}$ where $1 < m_1 \mid m_2 \mid \cdots \mid m_r$.

1. Prove that there exists an element of order m_r.
2. Show that an element of maximal order has order m_r.
3. Assume also that G is isomorphic to $C_{n_1} \times \ldots \times C_{n_s}$ with $1 < n_1 \mid \cdots \mid n_s$. Prove then that $r = s$ and $m_i = n_i$ for $i = 1, \ldots, r$.
   Hint: Use the fact that if G_1, G_2 and H are finite groups such that $G_1 \times H \cong G_2 \times H$, then $G_1 \cong G_2$ as well.

Exercise 2 (4 Points)**.**
Let G be the cyclic group of order n and let m be a positive integer dividing n. Show that G has a unique subgroup of order m.

Exercise 3 (10 Points)**.**
For $n \geq 2$, the symmetric group S_n is the group of permutations of the set $\{1, \ldots, n\}$. This is a finite group of order $n!$. Its elements can be expressed as a product of cycles on disjoint subsets of $\{1, \ldots, n\}$, and the expression is unique apart from the ordering of the cycles.

1. Show that the order of a permutation is the least common multiple of the lengths of the cycles in its cycle decomposition.
   Hint: Two disjoint cycles commute.
2. Does S_4 have elements of order 6?
3. Let σ be a cycle $(k_1 \ldots k_s)$ in S_n and $\tau \in S_n$. Show that $\tau\sigma\tau^{-1} = (\tau(k_1) \ldots \tau(k_s))$.
4. Let σ and ρ be conjugates, that is to say $\sigma = \tau^{-1}\rho\tau$ for some $\tau \in S_n$. Prove that σ and ρ have the same cycle decomposition type.
5. Prove that if two permutations have the same cycle decomposition type then they are conjugates.